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Note

The crossing number of $K_{2,m} \square P_n$ ☆

Zheng Wenping, Lin Xiaohui, Yang Yuansheng*

Department of Computer Science, Dalian University of Technology, Dalian, 116024, PR China

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Abstract

Investigation of the crossing number of graphs is a classical but very difficult problem. The exact value of the crossing number is known only for a few specific families of graphs. In this paper we extend a recent result from Bokal on the crossing number of $K_{1,m} \square P_n$, and prove that $cr(K_{2,m} \square P_n) = 2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$.

Keywords: Crossing number; Cartesian product; Bipartite graph; Path

1. Introduction

We consider only simple undirected graph and their good drawings.

Calculating the crossing number of a given graph is, in general, an elusive problem. Zarankiewicz [1] gave a drawing of $K_{r,m}$ which demonstrates that

$$\mathrm{cr}(K_{r,m}) \le Z(r,m) = \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

The equality holds for $\min(r,m) \le 6$ [2] and for the special cases $7 \le r \le 8$, $7 \le m \le 10$ [3]. The *Cartesian product* $G \square H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edge set $E(G \square H) = \{\{(x_1, y_1), (x_2, y_2)\} \mid x_1 = x_2 \text{ and } y_1 y_2 \in E(H) \text{ or } y_1 = y_2 \text{ and } x_1 x_2 \in E(G)\}$. There are several known exact results on the crossing numbers of Cartesian products of paths, cycles or stars with various other graphs [4–9]. In [7], Bokal proved that $\mathrm{cr}(K_{1,m} \square P_n) = (n-1)\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$.

In this paper, we extend Bokal's result, and prove that the crossing number of $K_{2,m} \square P_n$ is $2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$.

2. The upper bound of $\mathbf{cr}(K_{r,m} \square P_n)$ ($\min(r, m) \ge 2$)

Let

$$V(K_{r,m} \square P_n) = \bigcup_{i=0}^{n} (\{x_j^i \mid 0 \le j \le r-1\} \cup \{y_j^i \mid 0 \le j \le m-1\}),$$

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* Corresponding author.
E-mail address: yangys@dlut.edu.cn (Y. Yang).

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$$E(K_{r,m}\square P_n) = \left(\bigcup_{i=0}^{n}\{x_j^i y_k^i \mid 0 \le j \le r-1, 0 \le k \le m-1\}\right)$$
$$\cup \left(\bigcup_{i=1}^{n}\{x_j^{i-1} x_j^i \mid 0 \le j \le r-1\}\right) \cup \left(\bigcup_{i=1}^{n}\{y_j^{i-1} y_j^i \mid 0 \le j \le m-1\}\right).$$

For $0 \le i \le n$, let

$$V^i = \{x_j^i \mid 0 \le j \le r-1\} \cup \{y_j^i \mid 0 \le j \le m-1\},$$
$$E^i = \{x_j^i y_k^i \mid 0 \le j \le r-1, 0 \le k \le m-1\},$$
$$K_{r,m}^i = (V^i, E^i).$$

For $1 \le i \le n$, let

$$P_x^i = \{x_j^{i-1} x_j^i \mid 0 \le j \le r-1\},$$
$$P_y^i = \{y_j^{i-1} y_j^i \mid 0 \le j \le m-1\},$$
$$P^i = P_x^i \cup P_y^i.$$

Then, we have

$$E^i \cap E^j = \phi, \quad 0 \le i < j \le n,$$
$$P^i \cap P^j = \phi, \quad 1 \le i < j \le n,$$
$$E^i \cap P^j = \phi, \quad 0 \le i \le n, 1 \le j \le n,$$
$$E(K_{r,m}\square P_n) = \left(\bigcup_{i=0}^{n} E^i\right) \cup \left(\bigcup_{i=1}^{n} P^i\right).$$

Let A and B be two disjoint subsets of E. In a drawing D, the number of the crossings formed by an edge in A and another edge in B is denoted by $\nu_D(A, B)$. The number of the crossings that involve a pair of edges in A is denoted by $\nu_D(A)$. Then $\nu(D) = \nu_D(E)$. By counting the number of crossings in D, we have Lemma 2.1.

Lemma 2.1. *Let A, B, C be mutually disjoint subsets of E. Then,*

$$\nu_D(C, A \cup B) = \nu_D(C, A) + \nu_D(C, B),$$
$$\nu_D(A \cup B) = \nu_D(A) + \nu_D(B) + \nu_D(A, B).$$

Lemma 2.2. *For* $\min(r, m) \ge 2$*,*

$$\mathrm{cr}(K_{2,r,m}) \le Z(r+2, m+2) - rm$$

and

$$\mathrm{cr}(K_{1,r,m}) \le Z(r+1, m+1) - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. We will proof the theorem by exhibiting drawings in a cylinder. A cylinder can be 'assembled' from a polygon by identifying one pair of opposite sides of a rectangle [10].

Let

$$X = \{x_i \mid 0 \le i \le r-1\}, \qquad Y = \{y_j \mid 0 \le j \le m-1\}, \qquad Z = \{z_0, z_1\},$$
$$E_{XY} = \{x_i y_j \mid 0 \le i \le r-1, 0 \le j \le m-1\},$$
$$E_{XZ} = \{x_i z_j \mid 0 \le i \le r-1, 0 \le j \le 1\},$$
$$E_{YZ} = \{y_i z_j \mid 0 \le i \le m-1, 0 \le j \le 1\},$$
$$E(z_i) = \{z_i x_j \mid 0 \le j \le r-1\} \cup \{z_i y_j \mid 0 \le j \le m-1\}, \quad i = 0, 1,$$

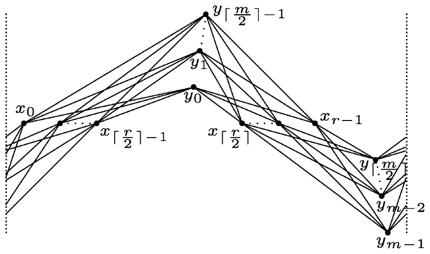

Fig. 2.1. A drawing of D_0 of $K_{r,m}$.

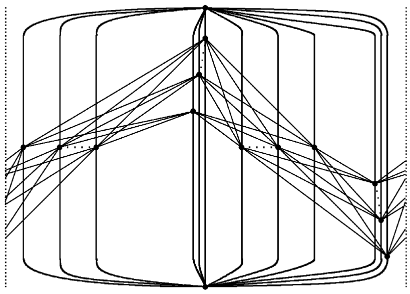

Fig. 2.2. A drawing of D_2 of $K_{2,r,m}$.

$$V(K_{r,m}) = X \cup Y, \qquad V(K_{2,r,m}) = X \cup Y \cup Z,$$
$$E(K_{r,m}) = E_{XY}, \qquad E(K_{2,r,m}) = E_{XY} \cup E_{XZ} \cup E_{YZ}.$$

A cylinder drawing D_2 of $K_{2,r,m}$ is exhibited in Fig. 2.2, where $\nu_{D_2}(E_{XZ} \cup E_{YZ}) = 0$ and $\nu(D_2) = \nu_{D_2}(E_{XY}) + \nu_{D_2}(E_{XY}, E_{XZ} \cup E_{YZ})$.

Let D_0 be the drawing of $K_{r,m}$ in D_2. Let $X_1 = \{x_i \mid 0 \le i \le \lceil \frac{r}{2} \rceil - 1\}$, $X_2 = \{x_i \mid \lceil \frac{r}{2} \rceil \le i \le r-1\}$, $Y_1 = \{y_i \mid 0 \le i \le \lceil \frac{m}{2} \rceil - 1\}$ and $Y_2 = \{y_i \mid \lceil \frac{m}{2} \rceil \le i \le m-1\}$. Now, for all $x \in X = X_1 \cup X_2$ and $y \in Y = Y_1 \cup Y_2$, join x and y by a line segment to form $K_{r,m}$. (We also show this construction in Fig. 2.1. Notice that for $x \in X_1$ and $y \in Y_2$, the line segment passes the identified sides.) For any pair vertices $x_{i,1}, x_{i,2} \in X_i$ $(i = 1, 2)$ and any pair vertices $y_{j,1}, y_{j,2} \in Y_j$ $(j = 1, 2)$, there is exactly one crossing, and hence the total number of crossings in D_0 is

$$\begin{aligned}\nu(D_0) = \nu_{D_2}(E_{XY}) &= \binom{\lfloor \frac{r}{2} \rfloor}{2}\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{r}{2} \rceil}{2}\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lfloor \frac{r}{2} \rfloor}{2}\binom{\lceil \frac{m}{2} \rceil}{2} + \binom{\lceil \frac{r}{2} \rceil}{2}\binom{\lceil \frac{m}{2} \rceil}{2} \\ &= \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor. \end{aligned} \tag{2.1}$$

For $0 \le i \le \lceil \frac{r}{2} \rceil - 1$, $z_0 x_i$ is crossed by all edges in $\{x_j y_k \mid i < j \le \lceil \frac{r}{2} \rceil - 1, \lceil \frac{m}{2} \rceil \le k \le m-1\}$. For $\lceil \frac{r}{2} \rceil \le i \le r-1$, $z_0 x_i$ is crossed by all edges in $\{x_j y_k \mid \lceil \frac{r}{2} \rceil \le j < i, \lceil \frac{m}{2} \rceil \le k \le m-1\}$. For $0 \le i \le \lceil \frac{m}{2} \rceil - 1$, $z_0 y_i$ is crossed by all edges in $\{y_j x_k \mid 0 \le j < i, \lceil \frac{r}{2} \rceil \le k \le r-1\}$. For $\lceil \frac{m}{2} \rceil \le i \le m-1$, $z_0 y_i$ is crossed by all edges in $\{y_j x_k \mid i < j \le m-1, \lceil \frac{r}{2} \rceil \le k \le r-1\}$. So the number of crossings on $E(z_0)$ is

$$\begin{aligned}\nu_{D_2}(E(z_0), E_{XY}) &= \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} \left(\left\lceil \frac{r}{2} \right\rceil - 1 - i \right) \left\lfloor \frac{m}{2} \right\rfloor + \sum_{i=\lceil \frac{r}{2} \rceil}^{r-1} \left(i - \left\lceil \frac{r}{2} \right\rceil \right) \left\lfloor \frac{m}{2} \right\rfloor \\ &\quad + \sum_{i=0}^{\lceil \frac{m}{2} \rceil - 1} i \left\lfloor \frac{n}{2} \right\rfloor + \sum_{i=\lceil \frac{m}{2} \rceil}^{m-1} (m-1-i) \left\lfloor \frac{r}{2} \right\rfloor \\ &= \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor. \end{aligned} \tag{2.2}$$

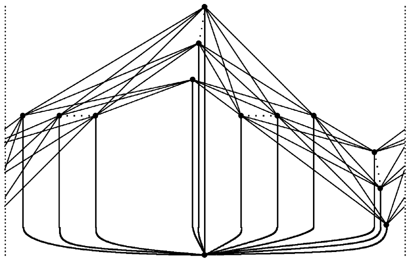

Fig. 2.3. A drawing of D_1 of $K_{1,r,m}$.

Similarly, the number of crossings on $E(z_1)$ is

$$\nu_{D_2}(E(z_1), E_{XY}) = \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{r+1}{2} \right\rfloor. \tag{2.3}$$

From (2.1)–(2.3), we have

$$\begin{aligned} \mathrm{cr}(K_{2,r,m}) \le \nu(D_2) &= \nu_{D_2}(E_{XY}) + \nu_{D_2}(E_{XY}, E_{XZ} \cup E_{YZ}) \\ &= \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor \\ &\quad + \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{r+1}{2} \right\rfloor \\ &= \left\lfloor \frac{r+2}{2} \right\rfloor \left\lfloor \frac{r+1}{2} \right\rfloor \left\lfloor \frac{m+2}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor - rm \\ &= Z(r+2, m+2) - rm. \end{aligned} \tag{2.4}$$

Deleting z_1 and all related edges in D_2, we obtain a drawing D_1 (Fig. 2.3) of $K_{1,m,n}$. From (2.3) and (2.4), we have

$$\begin{aligned} \mathrm{cr}(K_{1,r,m}) \le \nu(D_1) &= \nu(D_2) - \nu_{D_2}(E(z_1), E_{XY}) \\ &= \left\lfloor \frac{r+1}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \\ &= Z(r+1, m+1) - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor. \quad \square \end{aligned} \tag{2.5}$$

Theorem 2.1. *For* $\min(r, m) \ge 2$,

$$\mathrm{cr}(K_{r,m} \square P_n) \le (n-1)\left(Z(r+2, m+2) - rm\right) + 2\left(Z(r+1, m+1) - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor\right).$$

Proof. Fig. 2.4 shows a cylinder drawing D of $K_{r,m} \square P_n$ with $(n+1)$ disjoint $K_{r,m}$ and $(r+m)$ pairwise parallel n-paths. By counting the number of crossings in D, we obtain an upper bound of $\mathrm{cr}(K_{r,m} \square P_n)$, i.e.,

$$\begin{aligned} \mathrm{cr}(K_{r,m} \square P_n) \le \nu(D) &= \nu_D\left(\bigcup_{i=0}^{n} E^i \cup \bigcup_{i=1}^{n} P^i\right) \\ &= (n-1)\nu(D_2) + 2\nu(D_1) \\ &= (n-1)\left(Z(r+2, m+2) - rm\right) + 2\left(Z(r+1, m+1) - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor\right). \quad \square \end{aligned}$$

3. $\mathrm{cr}(K_{2,m} \square P_n) = 2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$

The vertex set of $K_{2,m} \square P_n$ is $\bigcup_{j=0}^{n}(X^j \cup Y^j)$, where $X^j := \{x_i^j \mid i = 0, 1\}$ and $Y^j := \{y_k^j \mid 0 \le k \le m-1\}$. The edge set is $(\bigcup_{j=0}^{n} E(X^j Y^j)) \cup (\bigcup_{j=0}^{n} E(X^j X^{j+1})) \cup (\bigcup_{j=0}^{n} E(Y^j Y^{j+1}))$, where $E(X^j Y^j) := \{x_i^j y_k^j \mid 0 \le i \le 1, 0 \le k \le m-1\}$, $E(X^j X^{j+1}) := \{x_i^j x_i^{j+1} \mid i = 0, 1\}$, $E(Y^j Y^{j+1}) := \{y_k^j y_k^{j+1} \mid 0 \le k \le m-1\}$.

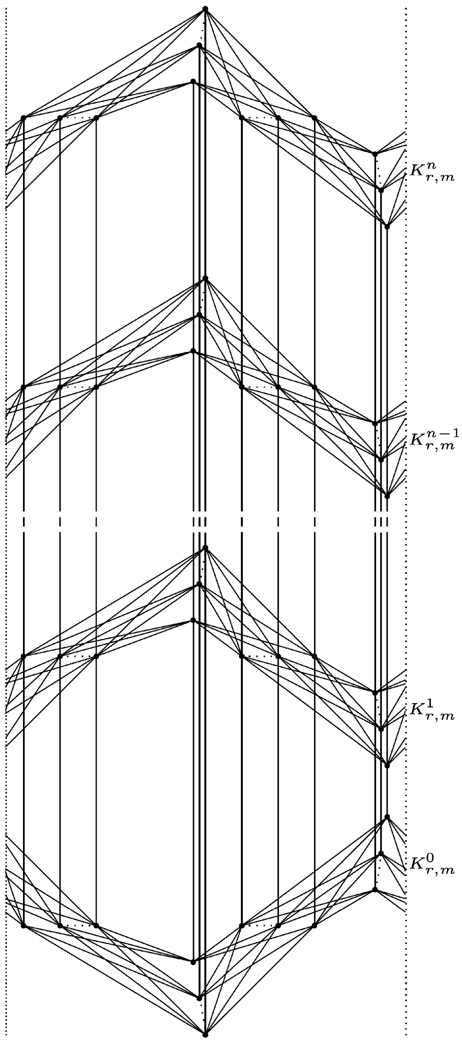

Fig. 2.4. A drawing of $K_{r,m}\Box P_n$.

Theorem 3.1. *The crossing number of* $K_{2,m}\Box P_n$ *is* $2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$.

Proof. Fig. 2.4 shows how to draw $K_{r,m}\Box P_n$ with $(n-1)(\lfloor\frac{m+2}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{r+2}{2}\rfloor\lfloor\frac{r+1}{2}\rfloor-mr)+2(\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{r+1}{2}\rfloor\lfloor\frac{r}{2}\rfloor-\lfloor\frac{m}{2}\rfloor\lfloor\frac{r}{2}\rfloor)$ crossings. For $r=2$, this gives $2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, and so $\mathrm{cr}(K_{2,m}\Box P_n)\le 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. To complete the proof we show the reverse inequality.

For each $j=1,2,\ldots,n-1$, and each $(s,t)\in\{0,1\}\times\{0,1\}$, let $T_{s,t}^{j}$ denote the subgraph of $K_{2,m}\Box P_n$ induced by x_s^{j-1}, X^j, x_t^{j+1}, Y^{j-1}, and Y^{j+1}, *minus* the edges $x_s^{j-1}x_s^{j}$ and $x_t^{j}x_t^{j+1}$. Each $T_{s,t}^{j}$ is homeomorphic to $K_{4,n}$. For $s=0,1$, let S_s^0 denote the subgraph of $K_{2,m}\Box P_n$ induced by x_s^{n-1}, X^n, Y^{n-1} and Y^n, minus $x_s^{n-1}x_s^n$. Each S_s^j is homeomorphic to $K_{3,n}$.

Consider any good drawing D of $K_{2,m}\Box P_n$. Let us say that a crossing in D among two edges in a $T_{s,t}^{j}$ (respectively, S_s^j) is *valid* if it is a good crossing even after we suppress the degree two vertices of $T_{s,t}^{j}$ (respectively, S_s^j).

Since $\mathrm{cr}(K_{3,m})=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\mathrm{cr}(K_{4,m})=2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, and there are $4(n-1)T_{s,t}^{j}$'s and 4 S_s^j's, it follows that the sum of the valid crossings over all the $T_{s,t}^{j}$'s is at least $4(n-1)\cdot 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, and the sum of the valid crossings over all the S_s^j's is at least $4\cdot\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$.

A quick inspection shows that (i) no valid crossing occurs both in a $T_{s,t}^{j}$ and in an S_s^j; (ii) each valid crossing occurs at most in four $T_{s,t}^{j}$'s; and (iii) each valid crossing occurs at most in two S_s^j's. Therefore the number of crossings in D is at least $(1/4)(4(n-1)\cdot 2\lfloor \frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor)+(1/2)(4\cdot\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor)=2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. $\square$

4. Conclusion

Furthermore, we have the following conjecture:

Conjecture 4.1.

$$\mathrm{cr}(K_{r,m}\square P_n)=(n-1)\left(Z(r+2,m+2)-rm\right)+2\left(Z(r+1,m+1)-\left\lfloor\frac{r}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\right).$$

From Bokal [7] *and Theorem* 3.1, *Conjecture* 4.1 *holds for* $\min(r,m)\le 2$.

Acknowledgements

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References

[1] K. Zarankiewicz, On a problem of P. Turán concerning graphs, Fund. Math 41 (1954) 137–145.
[2] D.J. Kleitman, The crossing number of $K_{5,n}$, J. Combin. Theory 9 (1971) 315–323.
[3] D.R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing-number conjecture, J. Graph Theory 17 (1993) 657–671.
[4] J. Adamsson, R.B. Richter, Arrangements, circular arrangements and the crossing number of $C_7\square C_n$, J. Combin. Theory Ser. B 90 (2004) 21–39.
[5] L.W. Beineke, R.D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, J. Graph Theory 4 (1980) 145–155.
[6] M. Klešč, A. Kocúrová, The crossing numbers of products of 5-vertex graphs with cycles, Discrete Math. 307 (2007) 1395–1403.
[7] D. Bokal, On the crossing numbers of Cartesian products with paths, J. Combin. Theory, B 97 (2007) 381–384.
[8] Zheng Wenping, Lin Xiaohui, Yang Yuansheng, Graphs Combin. 23 (2007) 327–336.
[9] S. Jendroľ, M. Ščerbová, On the crossing numbers of $S_m\square P_n$ and $S_m\square C_n$, Časopis pro Pěstování Matematiky 107 (1982) 225–230.
[10] L. Beineke, R. Wilson, Selected topics in graph theory, Academic Press, 1978, pp. 68–72.